Lectures on geometric realizations of crystals

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Abstract. These are notes for a lecture series given at the Fields Institute Summer School in Geometric Representation Theory and Extended Affine Lie Algebras, held at the University of Ottawa in June 2009. We give an introduction to the geometric realization of crystal graphs via the quiver varieties of Lusztig and Nakajima. The emphasis is on motivating the constructions through concrete examples. The relation between the geometric construction of crystals and combinatorial realizations using Young tableaux is also discussed.

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Introduction

These are notes for a lecture series given by the author at the Fields Institute Summer School in Geometric Representation Theory and Extended Affine Lie Algebras, held at the University of Ottawa in June 2009 and organized by the author and Erhard Neher. Each section corresponds to an 80 minute lecture given at the school.

The goal of these lectures is to give an introduction to the geometric realization of crystal graphs via quiver varieties. The aim throughout is to adequately motivate the definitions so that the reader gains an intuition for the constructions. For this reason, concrete examples are discussed in detail and mathematical rigor is sometimes sacrificed in the name of exposition. The hope is that after studying these lectures, the reader will have an intuitive grasp of the theory and several specific examples at hand that will equip him or her to explore the literature on this subject. References are given for proofs or arguments that have been omitted.

Section 1 is dedicated to motivating the definitions to follow in the other four lectures. Several examples are discussed in detail, noting connections to ideas that appeared in the lectures of Kamnitzer [6] and Kang [7] at the same summer school. Equipped with these examples, the discussion of the general theory begins in Section 2 where the notions of quivers and their representations are introduced. In Section 3, we define the Lusztig quiver varieties and the crystal structure on their sets of irreducible components. We see how one obtains the crystal corresponding to half of the quantized enveloping algebra of a symmetric Kac-Moody algebra. The lagrangian Nakajima quiver varieties are introduced in Section 4. Here we define the crystal structure on the sets of irreducible components and obtain the crystals corresponding to irreducible integrable highest weight representations. Finally, in Section 5 we describe the relationship between the geometric realizations of crystals using Nakajima quiver varieties and the well known combinatorial realizations using tableaux. In this final section we restrict our attention to the Lie algebra $\mathfrak{sl}_n$.

Prerequisites. In these notes, we assume a basic knowledge of Kac-Moody Lie algebras. In particular, we assume the reader is familiar with their definitions and the basics of the theory of irreducible integrable highest weight representations. We also presuppose a knowledge of the basic definitions of crystals. Students in the summer school benefited from a course on this subject [7]. A more comprehensive treatment can be found in the book [5]. Some background in algebraic geometry would be helpful in following these notes, but the reader willing to take some results on faith should be able to follow the presentation.

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Summer school lecture notes. These notes, along with lectures notes from the other speakers at the University of Ottawa Fields Institute Summer School on Geometric Representation Theory and Extended Affine Lie Algebras will eventually appear in the Fields Institute Monograph Series. Until then, notes (as well as video of the lectures) can be found at

`http://av.fields.utoronto.ca/video/08-09/geomrep/`.

1 Motivating examples

In this first section, we discuss some motivating examples for the theory that will be introduced in future sections. We will explicitly work out various special cases of the general objects we will introduce later (such as quiver varieties). Our goal is to create a collection of concrete examples that will guide our intuition and serve as motivation for the general definitions to follow. We begin by considering the following table.

Geometry	Algebra (Representation Theory)	Combinatorics (Crystals)
Varieties (components)	Vector space (basis)	Vertex set
Correspondences	Lie algebra action (Chevalley generators)	Crystal operators

In [6], Kamnitzer explained some relations between the first and second columns in this table. Namely he described certain varieties whose homology yielded the underlying vector space V of a representation of a Lie algebra $\mathfrak{g}$ and correspondences which produced operators realizing the action of Chevalley generators of $\mathfrak{g}$ on V. Then, in [7], Kang explained the passage from the second column to the third column. More precisely, he described how certain nice bases in representations of $U_q(\mathfrak{g})$ yield the vertices of a crystal graph in the $q \to 0$ limit and how the action of the Chevalley generators is replaced by crystal operators (colored directed edges of this graph).

In the current chapter, we will describe a general process in which one can pass from the first column directly to the third. In particular, for any (symmetric) Kac-Moody algebra, one can define what are called quiver varieties and from these we can obtain the crystal graph directly. The vertex set of the crystal is the set of irreducible components of the varieties and the crystal operators are given by natural geometric operators closely related to correspondences.

We begin by considering a specific example of the construction. Let $\mathfrak{g} = \mathfrak{sl}_n$, the Lie algebra of $n \times n$ traceless matrices. We let E_{ij} be the matrix with (i, j) entry equal to one and all other entries equal to zero. Then

$$\{e_k := E_{k,k+1},\ f_k := E_{k+1,k}\}_{1 \leq k \leq n-1}$$

are the *Chevalley generators* of $\mathfrak{g}$. We let $\mathfrak{h}$ be the *Cartan subalgebra* consisting of traceless diagonal matrices. Then $\mathfrak{h}$ has basis $\{h_k := E_{k,k} - E_{k+1,k+1}\}_{1 \leq k \leq n-1}$.

We define $\varepsilon_k \in \mathfrak{h}^*$, by $\varepsilon_k(E_{l,l}) = \delta_{k,l}$ for $1 \leq l, k \leq n$. Note that $\varepsilon_1 + \cdots + \varepsilon_n = 0$ since we consider traceless matrices. The dual space $\mathfrak{h}^*$ has a basis consisting of *simple roots* $\{\alpha_k := \varepsilon_k - \varepsilon_{k+1}\}_{1 \leq k \leq n-1}$. We also define the *fundamental weights* $\omega_k = \varepsilon_1 + \cdots + \varepsilon_k$, $1 \leq k \leq n-1$. Then

$$P := \bigoplus_{k=1}^{n-1} \mathbb{Z}\omega_i \quad \text{and} \quad P^+ := \bigoplus_{k=1}^{n-1} \mathbb{N}\omega_i$$

are the *weight lattice* and *dominant weight lattice* respectively. Elements of P^+ are called *dominant integral weights*. Any $w = w_1\omega_1 + \cdots + w_{n-1}\omega_{n-1} \in P^+$ can be

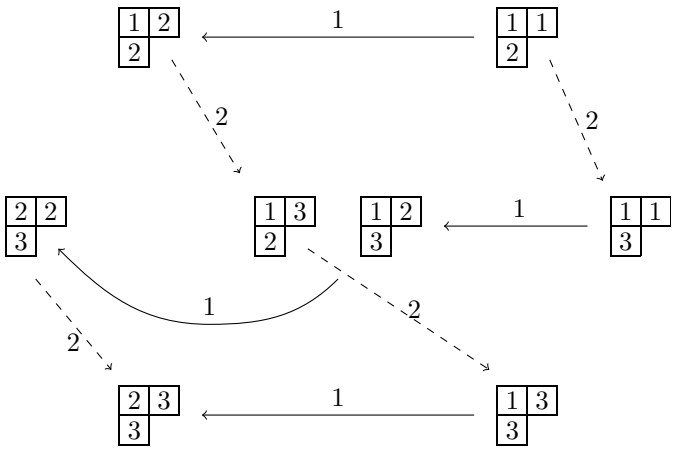


Figure 1 Crystal graph of $B((2,1))$, for $\mathfrak{g} = \mathfrak{sl}_3$.

written in the form

$$w = \lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1}, \quad \lambda_k = w_k + \cdots + w_{n-1}.$$

Thus w corresponds to a partition

$$\lambda(w) = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}), \quad \lambda_i \leq \lambda_{i+1},\ i = 1, \ldots, n-1.$$

In this way, we will often identify the set of partitions of length $n-1$ with the dominant weight lattice.

Finite-dimensional representations V of $\mathfrak{g}$ have a weight space decomposition

$$V = \bigoplus_{\mu \in P} V_\mu, \quad V_\mu = \{v \in V \mid h \cdot v = \mu(h)v \ \forall\ h \in \mathfrak{h}\}.$$

The action of $\mathfrak{g}$ on itself yields the *adjoint representation* and the corresponding weight space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \quad \dim \mathfrak{g}_\alpha = 1 \ \forall\ \alpha \in \Phi,\ \dim \mathfrak{h} = n-1,$$

is called the *root space decomposition* of $\mathfrak{g}$. Here $\Phi \subseteq P$ is the *set of roots* of $\mathfrak{g}$.

Irreducible representations of $\mathfrak{g}$ are labeled by their highest weight (an element of P^+). We denote the irreducible representation of $\mathfrak{g}$ of highest weight $w \in P^+$ by $V(w)$, or $V(\lambda(w))$ when we wish to label it by the corresponding partiion. The adjoint representation of $\mathfrak{g}$ is isomorphic to $V(\omega_1 + \omega_{n-1})$ as a $\mathfrak{g}$-module.

In [7, Section 4] Kang described a realization of the crystals of irreducible representations of $\mathfrak{g} = \mathfrak{sl}_n$ (or $\mathfrak{g} = \mathfrak{gl}_n$) via tableaux (see [5] for a more detailed description). Consider the example of $\mathfrak{g} = \mathfrak{sl}_3$ and $V = V(\omega_1 + \omega_2)$ the adjoint representation. Let $w = \omega_1 + \omega_2$ be the highest weight and let $\lambda = \lambda(w) = (2,1)$ be the corresponding partition. Then $B(\lambda)$ is the set of semistandard tableaux of shape λ and the crystal is as in Figure 1.

Recall that the weight of a tableaux T is equal to $\sum_{i=1}^n (\#\ i\text{'s in } T)\varepsilon_i$. In the above figure, the tableaux are grouped according to weight. The highest weight is $2\varepsilon_1 + \varepsilon_2 = \omega_1 + \omega_{n-1}$ and the lowest weight is $\varepsilon_2 + 2\varepsilon_3 = -\omega_1 - \omega_{n-1}$. All weight spaces are one-dimensional except for the $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ weight space $\mathfrak{h}$, which is

two dimensional. This corresponds to the fact that there are two tableaux (in the center of the figure) of this weight.

We now recall from [6, Section 4] Ginzburg's construction of irreducible representations of $\mathfrak{sl}_n$ (or $\mathfrak{gl}_n$) via Springer fibers. For comparison purposes, let us consider the same example. That is, let $\mathfrak{g} = \mathfrak{sl}_3$ and let $V = V(\omega_1 + \omega_2)$ be the adjoint representation as above. We set $w = \omega_1 + \omega_2$ and $\lambda = \lambda(w) = (2, 1)$. Recall that to construct the representation $V(w_1\omega_1 + \cdots + w_{n-1}\omega_{n-1})$ of $\mathfrak{sl}_n$, we fix a nilpotent endomorphism with w_i Jordan blocks of size i. So we fix a nilpotent $X \in \mathrm{End}(\mathbb{C}^3)$ with Jordan blocks of size 2 and 1.

Choose the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$ and take

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in this basis. Then the variety we are interested in is

$$\mathrm{Fl}_3(\mathbb{C}^3)^X = \{0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 = \mathbb{C}^3 \mid X(V_i) \subseteq V_{i-1},\ 1 \leq i \leq 3\},$$

and we have the decomposition

$$\mathrm{Fl}_3(\mathbb{C}^3)^X = \bigsqcup_{\mu \in \mathbb{N}^3,\, \mu_1+\mu_2+\mu_3=3} \mathrm{Fl}_\mu(\mathbb{C}^3)^X,$$

where $\mathrm{Fl}_\mu(\mathbb{C}^3)^X$ consists of the flags with $\dim V_i/V_{i-1} = \mu_i$.

Let $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 = \mathbb{C}^3$ be a flag in $\mathrm{Fl}_\mu(\mathbb{C}^3)^X$. Note that $X(V_3) = X(\mathbb{C}^3) = \langle e_1 \rangle$ and thus we must have $\langle e_1 \rangle \subseteq V_2$. Furthermore $\ker X = X^{-1}(0) = \langle e_1, e_3 \rangle$ and so $V_1 \subseteq \langle e_1, e_3 \rangle$. Now suppose that $\mu = (1, 1, 1)$. Thus $\dim V_i = i$ for $i = 0, 1, 2, 3$. We see that V_1 can be any one-dimensional subspace of $\langle e_1, e_3 \rangle$ and thus the choice of V_1 yields the projective line $\mathbb{P}^1$. Now, for a fixed V_1, consider the possibilities for V_2. We must have

$$V_1 + \langle e_1 \rangle = V_1 + X(V_3) \subseteq V_2 \subseteq X^{-1}(V_1).$$

Thus, if $V_1 = \langle e_1 \rangle$, the only conditions on V_2 are

$$\langle e_1 \rangle \subseteq V_2 \subseteq \mathbb{C}^3, \quad \dim V_2 = 2,$$

and so the choice of V_2 yields a projective line $\mathbb{P}^1$. On the other hand, if $V_1 \neq \langle e_1 \rangle$, then we must have $V_2 = \langle e_1 \rangle + V_1 = \langle e_1, e_3 \rangle$. Therefore, $\mathrm{Fl}_{(1,1,1)}(\mathbb{C}^3)^X$ consists of two $\mathbb{P}^1$'s meeting at a point.

Exercise 1.1 For all $\mu \in \mathbb{N}^3$ with $\mu_1 + \mu_2 + \mu_3 = 3$ and $\mu \neq (1, 1, 1)$, show that the variety $\mathrm{Fl}_\mu(\mathbb{C}^3)$ is either empty or is a point.

Recall that $H_{\mathrm{top}}(\mathrm{Fl}_3(\mathbb{C}^3)^X) \cong V(\lambda)$ and that the decomposition $H_{\mathrm{top}}(\mathrm{Fl}_3(\mathbb{C}^3)^X) = \bigoplus_\mu H_{\mathrm{top}}(\mathrm{Fl}_\mu(\mathbb{C}^3)^X)$ corresponds to the decomposition $V(\lambda) = \bigoplus_\mu V(\lambda)_\mu$. Furthermore, a basis of $H_{\mathrm{top}}(\mathrm{Fl}_\mu(\mathbb{C}^3)^X)$ is given by the fundamental classes of the irreducible components of $\mathrm{Fl}_\mu(\mathbb{C}^3)^X$. Using the above and Exercise 1.1, if we draw $\mathrm{Fl}_3(\mathbb{C}^3)^X$ we obtain Figure 2.

Note the similarity between Figures 1 and 2. Our goal is to form a crystal graph whose vertex set is the set of irreducible components of certain varieties (such as the Springer fibers discussed above) and whose crystal operators are defined geometrically. If we do this, we should be able to say which of the two $\mathbb{P}^1$'s in Figure 2 correspond to which of the weight zero tableaux in the center of Figure 1.

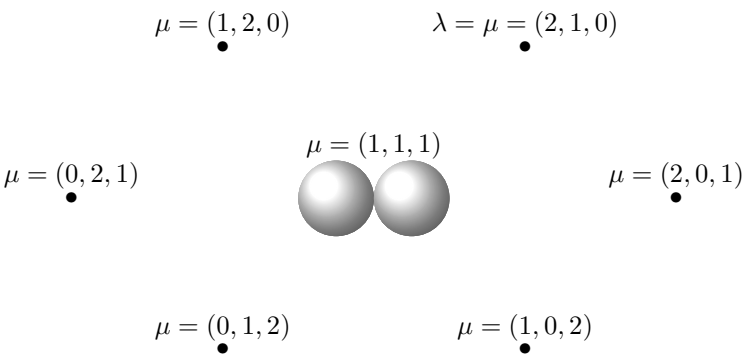


Figure 2 $\mathrm{Fl}_3(\mathbb{C}^3)^X$ where X has Jordan type $(2,1)$.

Let's look at another example. Consider $\mathfrak{g} = \mathfrak{sl}_n$ and the representation $V = V(N\omega_1)$. This corresponds to $X = 0$. Then

$$\mathrm{Fl}_\mu(\mathbb{C}^N)^X = \mathrm{Fl}_\mu(\mathbb{C}^N) = \{0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^N \mid \dim V_i/V_{i-1} = \mu_i\}$$

is irreducible for all μ. This corresponds to the fact that all weight spaces of V are one-dimensional. Therefore the set of irreducible components of $\mathrm{Fl}_n(\mathbb{C}^N)$ is precisely

$$\{\mathrm{Fl}_\mu(\mathbb{C}^N) \mid \mu \in \mathbb{N}^n,\ \mu_1 + \cdots + \mu_n = N\}.$$

We wish to construct crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ in a "natural" geometric way. For $i = 1, \ldots, n-1$, define

$$\mu \pm \alpha_i = (\mu_1, \ldots, \mu_{i-1}, \mu_i \pm 1, \mu_{i+1} \mp 1, \mu_{i+1}, \ldots, \mu_n).$$

Define

$$\mathrm{Fl}_{\mu,\mu+\alpha_i}(\mathbb{C}^N) = \{((U_j),(V_j)) \in \mathrm{Fl}_\mu(\mathbb{C}^N) \times \mathrm{Fl}_{\mu+\alpha_i}(\mathbb{C}^N) \mid U_j \subseteq V_j\ \forall\ j\}.$$

Note that the condition $U_j \subseteq V_j$ implies that $U_j = V_j$ for all $j \neq i$. Then we have natural projections

$$\mathrm{Fl}_\mu(\mathbb{C}^N) \xleftarrow{\pi_1} \mathrm{Fl}_{\mu,\mu+\alpha_i}(\mathbb{C}^N) \xrightarrow{\pi_2} \mathrm{Fl}_{\mu+\alpha_i}(\mathbb{C}^N)$$

given by $\pi((U_j),(V_j)) = (U_j)$ and $\pi_2((U_j),(V_j)) = (V_j)$.

What are the fibers of the maps π_1 and π_2? Fix $(U_j) \in \mathrm{Fl}_\mu(\mathbb{C}^N)$. Then

$$\begin{aligned}\pi_1^{-1}((U_j)) &\cong \{V_i \mid U_i \subseteq V_i \subseteq U_{i+1},\ \dim V_i/U_i = 1\} \\ &\cong \{\bar{V} \subseteq U_{i+1}/U_i \mid \dim \bar{V} = 1\} \\ &\cong \mathbb{P}^{\mu_{i+1}-1}.\end{aligned}$$

So π_1 is a fiber bundle with smooth fibers isomorphic to $\mathbb{P}^{\mu_{i+1}-1}$.

Exercise 1.2 Show that π_2 is a fiber bundle with smooth fibers isomorphic to $\mathbb{P}^{\mu_i-1}$.

Since π_1 and π_2 are both fiber bundles with smooth fibers, they induce the following bijections (provided $\mu + \alpha_i \in \mathbb{N}^n$).

$$\begin{array}{ccccc} \text{Irreducible components of } \mathrm{Fl}_{\mu}(\mathbb{C}^N) & \overset{\text{1-1}}{\longleftrightarrow} & \text{Irreducible components of } \mathrm{Fl}_{\mu,\mu+\alpha_i}(\mathbb{C}^N) & \overset{\text{1-1}}{\longleftrightarrow} & \text{Irreducible components of } \mathrm{Fl}_{\mu+\alpha_i}(\mathbb{C}^N) \end{array}$$

We can then use these bijections to define the actions of the crystal operators $\tilde{e}_i$ and $\tilde{f}_i$, $i = 1, \ldots, n-1$. Namely, $\tilde{e}_i$ sends an irreducible component of $\mathrm{Fl}_{\mu}(\mathbb{C}^N)$ to the corresponding irreducible component of $\mathrm{Fl}_{\mu+\alpha_i}(\mathbb{C}^N)$ and $\tilde{f}_i$ does the opposite.

This example will motivate the more general construction in Section 4. However, we will need to do some extra work because the above example was a bit too simple. In particular, the weight spaces were all one-dimensional and so the varieties involved were all irreducible. We will need to deal with representations whose weight spaces have higher dimension and thus whose corresponding varieties are not irreducible. This will require us to develop slightly more sophisticated operators.

2 Quivers

The varieties we will use in our geometric construction of crystal bases are certain varieties attached to quivers. In this section we review some of the basic theory of quivers and their representations.

A *quiver* is simply another name for a directed graph. Thus, a quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are sets and s and t are maps from Q_1 to Q_0. We call Q_0 and Q_1 the sets of *vertices* and *directed edges* (or *arrows*) respectively. The maps s and t tell us the endpoints of each arrow: for an arrow $a \in Q_1$, we call $s(a)$ the *source* of a and $t(a)$ the *target* of a. Usually we will write $Q = (Q_0, Q_1)$, leaving the maps s and t implied. The quiver Q is said to be *finite* if Q_0 and Q_1 are both finite. A *loop* is an arrow a with $s(a) = t(a)$. In this paper, all quivers will be assumed to be finite and without loops. If we forget the orientation of the edges in a quiver, we obtain a graph, called the *underlying graph* of the quiver. A quiver is said to be of *finite type* its underlying graph is a Dynkin diagram of finite ADE type. Similarly, it is of *affine* (or *tame*) *type* if the underlying graph is a Dynkin diagram of affine type and of *indefinite* (or *wild*) *type* if the underlying graph is a Dynkin diagram of indefinite type.

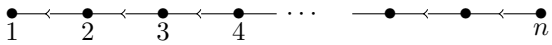
A *path* in Q is a sequence of arrows lining up tip-to-tail. More precisely, a path is a sequence $\beta = a_l a_{l-1} \cdots a_1$ of arrows such that $t(a_i) = s(a_{i+1})$ for $1 \leq i \leq l-1$. We call l the *length* of the path. We let $s(\beta) = s(a_1)$ and $t(\beta) = t(a_l)$ denote the initial and final vertices of the path β. For each vertex $i \in I$, we have a trivial path e_i with $s(e_i) = t(e_i) = i$.

The *path algebra* $\mathbb{C}Q$ associated to a quiver Q is the $\mathbb{C}$-algebra whose underlying vector space has basis the set of paths in Q, and with the product of paths given by concatenation. More precisely, if $\beta = a_l \cdots a_1$ and $\beta' = b_m \cdots b_1$ are two paths in Q, then $\beta\beta' = a_l \cdots a_1 b_m \cdots b_1$ if $t(\beta') = s(\beta)$ and $\beta\beta' = 0$ otherwise. This is an associative multiplication. There is a natural grading $\mathbb{C}Q = \bigoplus_{n \in \mathbb{N}} (\mathbb{C}Q)_n$ where $(\mathbb{C}Q)_n$ is the span of the paths of length n. This is a grading as an algebra in the sense that $(\mathbb{C}Q)_n (\mathbb{C}Q)_m \subseteq (\mathbb{C}Q)_{n+m}$ (in fact, we have equality here).

Example 2.1 Let Q be the *Jordan quiver*:

Then $\mathbb{C}Q = \mathbb{C}[t]$, the polynomial algebra in one variable. The variable t corresponds to the unique path of length one.

Exercise 2.2 Let Q be the following *quiver of type* A_n:



Then for all $1 \leq i \leq j \leq n$, there exists a unique path p_{ij} from j to i. Define a map $f : \mathbb{C}Q \to M_{n\times n}(\mathbb{C})$, the algebra of $n \times n$ matrices with complex entries, by $p_{ij} \mapsto E_{ij}$ and extending by linearity. Show that f is an isomorphism (of algebras) onto the algebra of upper triangular matrices.

A *representation* (V, x) of a quiver Q is a collection

$$\{V_i \mid i \in Q_0\}$$

of finite-dimensional vector spaces, together with a collection

$$\{x_a : V_{s(a)} \to V_{t(a)} \mid a \in Q_1\}$$

of linear maps. Representations of Q are equivalent to representations of the path algebra $\mathbb{C}Q$.

We will often view the collection $\{V_i \mid i \in Q_0\}$ as an Q_0-graded vector space $V = \bigoplus_{i\in Q_0} V_i$. Then the *graded dimension* of V is

$$\mathbf{dim}\, V = (\dim V_i)_{i\in Q_0}.$$

A morphism $(V, x) \to (W, y)$ between two representations of a quiver Q is a collection

$$\{\psi_i : V_i \to W_i \mid i \in Q_0\}$$

of linear maps such that the diagram

$$\begin{array}{ccc} V_{s(a)} & \xrightarrow{x_a} & V_{t(a)} \\ {\scriptstyle \psi_{s(a)}}\downarrow & & \downarrow{\scriptstyle \psi_{t(a)}} \\ W_{s(a)} & \xrightarrow{y_a} & W_{t(a)} \end{array}$$

commutes for all $a \in Q_1$.

Quivers and path algebras play an important role in the representation theory of finite-dimensional associative algebras. In particular, we have the following result.

Proposition 2.3 *Let A be a finite-dimensional $\mathbb{C}$-algebra. Then the category of representations of A is equivalent to the category of representations of $\mathbb{C}Q/I$ for some quiver Q and some two-sided ideal I of $\mathbb{C}Q$.*

Proof This follows from results in [3, 4]. The full proof can also be found in [1, Theorem 3.7]. □

Suppose (V, x) and (W, y) are representations of a quiver Q. Then we define the *direct sum* of (V, x) and (W, y) to be the representation $(V \oplus W, x \oplus y)$ where $(V \oplus W)_i = V_i \oplus W_i$ for $i \in Q_0$, and $(x \oplus y)_a = x_a \oplus y_a$ for $a \in Q_1$.

If (V, x) is a representation of a quiver Q and W is an Q_0-graded subspace of V (that is, $W_i \subseteq V_i$ for all $i \in Q_0$) then we say that W is *x-invariant* if $x_a(W_{s(a)}) \subseteq W_{t(a)}$ for all $a \in Q_1$. Then $(W, x|_W)$ is called a *subrepresentation* of (V, x). The representation $(0, 0)$ is called the *trivial representation*.

A quiver representation Q is said to be *irreducible* (or *simple*) if it contains no nontrivial proper subrepresentations. It is called *indecomposable* if cannot be written as a direct sum of two nontrivial subrepresentations.

Example 2.4 Fix a vertex $i \in Q_1$ and let S^i be a Q_0-graded vector space with $S^i_i = \mathbb{C}$ and $S^i_j = 0$ for $i \neq j$. Then $(S^i, 0)$ is a simple representation.

Example 2.5 Let Q be the quiver of Exercise 2.2. For $1 \leq i \leq j \leq n$, define $V^{i,j}$ to be the Q_0-graded vector space defined by

$$V^{i,j}_k = \begin{cases} \mathbb{C} & i \leq k \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

For $a \in Q_1$, define

$$x^{i,j}_a = \begin{cases} \mathrm{id} & i+1 \leq s(a) \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(V^{i,j}, x^{i,j})$ is indecomposable. However, if $i < j$, then $(V^{i,j}, x^{i,j})$ is not irreducible.

Forgetting the orientation of the edges of a quiver Q, we obtain the *underlying graph* $\Gamma(Q)$ of Q. We then let $\mathfrak{g}(Q)$ be the Kac-Moody algebra whose Dynkin graph is the graph $\Gamma(Q)$. Note that such a Kac-Moody algebra has symmetric Cartan matrix and any Kac-Moody algebra with symmetric Cartan matrix arises as $\mathfrak{g}(Q)$ for some (in general not unique) quiver Q.

Theorem 2.6 (Gabriel's Theorem [2, 3]) *A quiver Q has finitely many indecomposable representations (up to isomorphism) if and only if Q is of finite type. Furthermore, if Q is of finite type, there is a one-to-one correspondence between isomorphism classes of indecomposable representations of Q and positive roots of $\mathfrak{g}(Q)$. This correspondence is given by*

$$(V, x) \mapsto \sum_{i \in Q_0} (\dim V_i)\alpha_i,$$

where the α_i, $i \in Q_0$, are the simple roots of $\mathfrak{g}(Q)$.

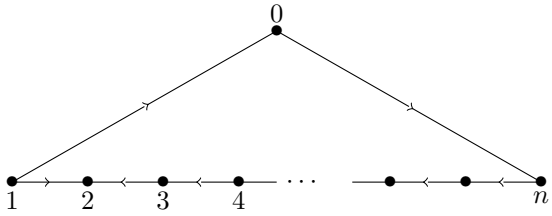
Proof This result was first proved by Gabriel in [3]. The proof is computational and the relation with Dynkin diagrams is a consequence of this proof and not an important feature of the calculations. A second proof was given by Bernšteĭn, Gel′fand, and Ponomarev in [2] that involves the machinery of Coxeter functors and Weyl groups. □

Exercise 2.7 (Type A_n) Let Q be the quiver of type A_n given in Exercise 2.2. Then $\mathfrak{g}(Q) = \mathfrak{sl}_{n+1}$. Show that the representations $(V^{i,j}, x^{i,j})$ are the only indecomposable representations of Q up to isomorphism. The indecomposable representation $(V^{i,j}, x^{i,j})$ corresponds to the positive root

$$\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$$

which is the weight of the root space $\mathbb{C}E_{i,j+1}$ of $\mathfrak{sl}_{n+1}$, where $E_{i,j+1}$ is the elementary matrix with a one in entry $(i, j+1)$ and a zero in all other entries.

Example 2.8 (Type $A^{(1)}_n$) Let Q be the quiver



and consider the positive imaginary root $\delta = \sum_{i=0}^{n} \alpha_i$. Then there exists a one parameter family of non-isomorphic representations where $\dim V_i = 1$ for $i = 0, \ldots, n$, and all x_a are non-zero. The parameter is the composition around the loop.

Theorem 2.9 (Kac's Theorem) *Let Q be an arbitrary quiver. Then the graded dimensions of indecomposable representations of Q correspond, via the map*

$$\mathbf{dim}\, V \mapsto \sum_{i \in Q_0} (\dim V_i)\alpha_i,$$

to the positive roots of the root system of $\mathfrak{g}(Q)$.

Remarks 2.10
1. In Kac's Theorem, no assertion is made that positive roots are in one-to-one correspondence with isomorphism classes of indecomposable representations.
2. Real roots correspond to graded dimensions with one indecomposable representation of that dimension and imaginary roots correspond to graded dimensions with families of indecomposable representations of that dimension.

Assume for simplicity that Q is of finite type. Then any representation can be written in a unique way (up to isomorphism) as a sum of indecomposable representations labeled by positive roots of $\mathfrak{g} = \mathfrak{g}(Q)$. Let Φ be the set of roots of $\mathfrak{g}$ and $\Phi^{\pm}$ the set of positive/negative roots. Recall that $\mathfrak{g}$ has a *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{n}^- = \bigoplus\nolimits_{\beta \in \Phi^-} \mathfrak{g}_\beta, \quad \mathfrak{g}_\beta = \mathbb{C} f_\beta, \quad \beta \in \Phi^-.$$

We fix an ordering $\beta_1, \ldots, \beta_m$ of Φ^-. Then, by the PBW Theorem, $U(\mathfrak{n}^-)$ has a basis

$$\{f_{\beta_1}^{l_1} f_{\beta_2}^{l_2} \ldots f_{\beta_m}^{l_m} \mid l_i \in \mathbb{N},\ 1 \leq i \leq m\}.$$

Thus we have a bijection

$$\{\text{Isomorphism classes of representations of } Q\} \overset{\text{1-1}}{\longleftrightarrow} \text{Basis of } U(\mathfrak{g}^-).$$

Fix a Q_0-graded vector space $V = \bigoplus_{i \in Q_0} V_i$ and let $\mathbf{v} = \mathbf{dim}\, V$. Then define $A_{\mathbf{v}}$ to be the space of representations of Q on V. More precisely,

$$A_{\mathbf{v}} := \{x = (x_a)_{a \in Q_1} \mid x_a : V_{s(a)} \to V_{t(a)},\ a \in Q_1\}.$$

Let $G = G_V = \prod_{i \in Q_0} GL(V_i)$. Then G acts on $A_{\mathbf{v}}$ by

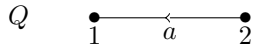
$$g \cdot x = (g_i)_{i \in Q_0} \cdot (x_a)_{a \in Q_1} = (x'_a)_{a \in Q_1}, \quad x'_a = g_{t(a)} x_a g_{s(a)}^{-1}.$$

In this way, the G-orbits of $A_{\mathbf{v}}$ are precisely the isomorphism classes of representations of Q with graded dimension $\mathbf{v}$. Therefore, we have a bijection

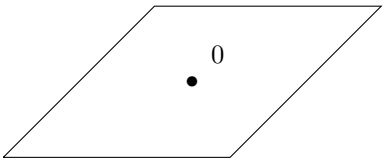
$$\{G\text{-orbits of } A_{\mathbf{v}}\} \overset{\text{1-1}}{\longleftrightarrow} \text{Basis of } U(\mathfrak{n}^-)_{-\sum_i v_i \alpha_i},$$

where $U(\mathfrak{n}^-)_\beta$ denotes the β-weight space of $U(\mathfrak{n}^-)$.

Example 2.11 Consider the quiver



Then $\mathfrak{g}(Q) = \mathfrak{sl}_3$. Let V be the Q_0-graded vector space given by $V_1 = \mathbb{C}$ and $V_2 = \mathbb{C}$ and let $\mathbf{v} = \mathbf{dim}\, V$. Then $A_{\mathbf{v}}$ is the space of linear maps from $\mathbb{C}$ to $\mathbb{C}$, which is isomorphic to $\mathbb{C}$ (since any such map is given by multiplication by some complex number). Then $A_{\mathbf{v}} \cong \mathbb{C}$ decomposes as a disjoint union of two orbits: $\mathbb{C}^*$ and $\{0\}$.



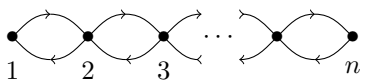
We would like to construct a variety whose irreducible components are in one-to-one correspondence with the elements of a basis of a weight space of $U(\mathfrak{n}^-)$. The variety $A_{\mathbf{v}}$ does not satisfy this property because it has only one irreducible component. The problem is that its orbits fit together in such a way as to form a single irreducible component. We would like each orbit to give rise to its own irreducible component. One way of doing this would be to take the union of the conormal bundles to the orbits in $A_{\mathbf{v}}$.

We recall the definition of the conormal bundle. The conormal bundle to a smooth subvariety S of a smooth variety A is the sub-bundle of the cotangent bundle T^*A whose fiber over any point $x \in S$ consists of those $\phi \in (T^*A)_x$ such that $\phi(v) = 0$ for all $v \in (TS)_x$ and whose fiber over all other points is empty. The dimension of the conormal bundle is equal to the dimension of A (and half the dimension of the cotangent bundle T^*A). The larger the dimension of the orbit, the smaller the dimension of the fibers.

We thus see that the union of the conormal bundles to the orbits in $A_{\mathbf{v}}$ has the property we desire. Namely, its set of irreducible components is in one-to-one correspondence with a basis of the $-\sum_i v_i\alpha_i$ weight space of $U(\mathfrak{n}^-)$. In particular, the conormal bundle to the orbit $\mathbb{C}^*$ is simply the orbit itself while the conormal bundle to the orbit $\{0\}$ is a complex line. We will see in the next section that this observation leads naturally to a "doubling" of the quiver and the definition of the Lusztig quiver variety.

3 The Lusztig quiver variety

Let $\mathfrak{g}$ be a Kac-Moody algebra with symmetric Cartan matrix. For example, $\mathfrak{g}$ could be a simple Lie algebra of type A, D or E. We let $Q = (Q_0, Q_1)$ be the *double quiver* associated to the Dynkin graph of $\mathfrak{g}$. That is, Q_0 is the set of vertices of this Dynkin graph and for each (undirected) edge of the Dynkin graph, Q_1 contains two arrows (one in each direction) with the same endpoints. For example, if $\mathfrak{g} = \mathfrak{sl}_{n+1}$ is the Lie algebra of type A_n, then the corresponding double quiver is as follows.



We have a natural involution $a \mapsto \bar{a}$, $a \in Q_1$, which maps the arrow a to the arrow with the same underlying edge but with opposite orientation.

An *orientation* of Q is a choice of subset $\Omega \subseteq Q_1$ such that $\Omega \cup \bar{\Omega} = Q_1$ and $\Omega \cap \bar{\Omega} = \emptyset$. That is, Ω contains exactly one arrow from each pair associated to each edge of the Dynkin graph.

For a Q_0-graded vector space $V = \bigoplus_{i \in Q_0} V_i$, let

$$E_V = \bigoplus_{a \in Q_1} \operatorname{Hom}(V_{s(a)}, V_{t(a)}).$$

Then $E_V = E_{V,\Omega} \oplus E_{V,\bar{\Omega}}$, where

$$E_{V,\Omega} = \bigoplus_{a \in \Omega} \operatorname{Hom}(V_{s(a)}, V_{t(a)}), \quad E_{V,\bar{\Omega}} = \bigoplus_{a \in \bar{\Omega}} \operatorname{Hom}(V_{s(a)}, V_{t(a)}).$$

Recall that $G_V = \prod_{i \in Q_0} GL(V_i)$ acts naturally on E_V, $E_{V,\Omega}$ and $E_{V,\bar{\Omega}}$.

Define

$$\epsilon : Q_1 \to \{\pm 1\}, \quad \epsilon(a) = \begin{cases} +1 & a \in \Omega, \\ -1 & a \in \bar{\Omega}. \end{cases}$$

We then define a G_V-invariant, nondegenerate, symplectic form $\langle \cdot, \cdot \rangle$ on E_V by

$$\langle x, y \rangle = \sum_{a \in Q_1} \epsilon(a) \operatorname{tr}(x_a x_{\bar{a}}).$$

This pairs $E_{V,\Omega}$ with $E_{V,\bar{\Omega}}$ and so we can view $E_{V,\bar{\Omega}}$ as the dual space $(E_{V,\Omega})^*$ and E_V as the tangent space $T^* E_{V,\Omega}$ to $E_{V,\Omega}$. This is simply the observation that $\operatorname{Hom}(V, W)$ is dual to $\operatorname{Hom}(W, V)$ under the trace.

To any Hamiltonian action of a Lie group on a symplectic manifold, there is an associated *moment map*. The G_V-action on E_V is such an action and the corresponding moment map is

$$\psi : E_V \to \mathfrak{gl}_V = \prod_{i \in Q_0} \mathfrak{gl}(V_i) = \prod_{i \in Q_0} \operatorname{End} V_i,$$

with ith component

$$\psi_i(x) = \sum_{a \in Q_1,\, t(a) = i} \epsilon(a) x_a x_{\bar{a}}.$$

Here $\mathfrak{gl}_V$ is the Lie algebra of G_V. Usually the moment map is a map to the dual of the Lie algebra of the group but we have identified $\mathfrak{gl}_V$ with its dual via the trace here.

We say that $x \in E_V$ is *nilpotent* if there exists an $N \geq 1$ such that for any path $\beta = a_N \cdots a_2 a_1$ of length N, we have that

$$x_{a_N} \cdots x_{a_2} x_{a_1} : V_{s(a_1)} \to V_{t(a_N)}$$

is the zero map.

Definition 3.1 (Lusztig quiver variety) For a Q_0-graded vector space $V = \bigoplus_{i \in Q_0} V_i$, let

$$\Lambda_V := \{x \in E_V \mid \psi(x) = 0,\ x \text{ is nilpotent}\}.$$

The variety Λ_V is called the *Lusztig quiver variety* associated to Q and V.

For $\mathfrak{g}$ (or Q) of arbitrary type, Λ_V has the following properties:

1. Λ_V is a closed subvariety of E_V of pure dimension $\frac{1}{2} \dim E_V$. That is, each irreducible component of Λ_V has this dimension.
2. Λ_V is a langrangian subvariety of E_V.
3. If $x_\Omega \in E_{V,\Omega}$ and $x_{\bar{\Omega}} \in E_{V,\bar{\Omega}}$, then

$$\psi(x_\Omega + x_{\bar{\Omega}}) = 0 \quad \Longleftrightarrow \quad \begin{array}{l} x_{\bar{\Omega}} \text{ is orthogonal to the tangent space to the } G_V\text{-} \\ \text{orbit through } x_\Omega \text{ (with respect to the form } \langle \cdot, \cdot \rangle). \end{array}$$

Additionally, if $\mathfrak{g}$ is of finite type, then

4. If $x \in E_V$, then $\psi(x) = 0$ implies that x is nilpotent. Thus, the nilpotency condition in the definition of the Lusztig quiver variety is superfluous.
5. The irreducible components of Λ_V are the closures of the conormal bundles of the G_V-orbits in $E_{V,\Omega}$.

Our goal is to use Lusztig quiver varieties to construct the crystal $B(\infty)$. We want the following relationship between the geometry and the elements of the crystal.

Crystal	Geometry
Vertex set	Irreducible components of $\bigsqcup_V \Lambda_V$
Vertices of weight $-\sum v_i \alpha_i$	Irreducible components of Λ_V, $\mathbf{dim}\, V = \mathbf{v}$
Crystal operators	Natural geometrically defined operators

In the union $\bigsqcup_V \Lambda_V$, we take the union over one Q_0-graded vector space V of each graded dimension.

We now describe this process in more detail. For $\mathbf{v} \in \mathbb{N}^{Q_0}$, set $V^{\mathbf{v}} = \bigoplus_{i \in Q_0} \mathbb{C}^{v_i}$ and $\Lambda(\mathbf{v}) = \Lambda_{V^{\mathbf{v}}}$. For $i \in Q_0$, let $\mathbf{e}^i \in \mathbb{N}^{Q_0}$ such that $\mathbf{e}^i_j = \delta_{ij}$ and for $c \in \mathbb{N}$, define $\tilde{\Lambda}(\mathbf{v}, c\mathbf{e}^i)$ to be the variety of triples $(x, \phi', \bar{\phi})$ where $x \in \Lambda(\mathbf{v})$ and $\phi' = (\phi'_i)_{i \in I}$, $\bar{\phi} = (\bar{\phi}_i)_{i \in I}$ give an exact sequence

$$0 \to V^{\mathbf{v} - c\mathbf{e}^i} \xrightarrow{\phi'} V^{\mathbf{v}} \xrightarrow{\bar{\phi}} V^{c\mathbf{e}^i} \to 0$$

such that $\operatorname{im} \phi'$ is x-invariant. Then x induces $x' \in \Lambda(\mathbf{v} - c\mathbf{e}^i)$ by the restriction to $\operatorname{im} \phi'$ and $\bar{x} \in \Lambda(c\mathbf{e}^i) = \{0\}$ by passing to the quotient $V^{\mathbf{v}} / \operatorname{im} \phi'$. Note that x is nilpotent if and only if x' is.

Consider the maps

$$\Lambda(\mathbf{v} - c\mathbf{e}^i) \xleftarrow{p_1} \tilde{\Lambda}(\mathbf{v}, c\mathbf{e}^i) \xrightarrow{p_2} \Lambda(\mathbf{v})$$

where $p_1(x, \phi', \bar{\phi}) = x'$ and $p_2(x, \phi', \bar{\phi}) = x$. We want to use these maps to identify irreducible components. The problem is that they are not the right type of maps (that is, fiber bundles with smooth fibers). Thus, we need to restrict them.

For $i \in Q_0$, define $\varepsilon_i : \Lambda(\mathbf{v}) \to \mathbb{N}$ by

$$\varepsilon_i(x) = \dim \operatorname{Coker} \left(\bigoplus_{a,\, t(a) = i} V_{s(a)} \xrightarrow{(x_a)} V_i \right).$$

We will see that this map will play the role of the map ε_i in the definition of crystals (see [7, Section 5]), hence the notation. For $c \in \mathbb{N}$, define

$$\Lambda(\mathbf{v})_{i,c} = \{x \in \Lambda(\mathbf{v}) \mid \varepsilon_i(x) = c\}.$$

This is a locally closed subvariety of $\Lambda(\mathbf{v})$.

If $\Lambda(\mathbf{v})_{i,c} \neq \emptyset$, then

$$p_1^{-1}(\Lambda(\mathbf{v} - c\mathbf{e}^i)_{i,0}) = p_2^{-1}(\Lambda(\mathbf{v})_{i,c}) := \tilde{\Lambda}(\mathbf{v}, c\mathbf{e}^i)_{i,0}$$

and we have

$$\Lambda(\mathbf{v} - c\mathbf{e}^i)_{i,0} \xleftarrow{p_1} \tilde{\Lambda}(\mathbf{v}, c\mathbf{e}^i)_{i,0} \xrightarrow{p_2} \Lambda(\mathbf{v})_{i,c}. \tag{3.1}$$

Lemma 3.2 ([8, Lemma 5.2.3]) *The maps p_1 and p_2 in* (3.1) *are fiber bundles with smooth fibers.*

Exercise 3.3 Describe the fibers (see [8, Lemma 5.2.3]).

Corollary 3.4 *If $\Lambda(\mathbf{v})_{i,c} \neq \emptyset$, then we have a bijective correspondence*

$$\textit{Irreducible components of } \Lambda(\mathbf{v}-c\mathbf{e}^i)_{i,0} \quad \overset{1\text{-}1}{\longleftrightarrow} \quad \textit{Irreducible components of } \Lambda(\mathbf{v})_{i,c}.$$

Let $B(\mathbf{v},\infty)$ be the set of irreducible components of $\Lambda(\mathbf{v})$ and $B_g(\infty) = \bigsqcup_{\mathbf{v}} B(\mathbf{v},\infty)$. Since the sets $\Lambda(\mathbf{v})_{i,c}$ are locally closed, for each $X \in B(\mathbf{v},\infty)$ there is an open dense subset of X where ε_i takes a fixed value. We define $\varepsilon_i(X)$ to be this value. For $c \in \mathbb{N}$, we let

$$B(\mathbf{v},\infty)_{i,c} = \{X \in B(\mathbf{v},\infty) \mid \varepsilon_i(X) = c\}.$$

By Corollary 3.4, we have a bijective correspondence

$$B(\mathbf{v}-c\mathbf{e}^i)_{i,0} \cong B(\mathbf{v})_{i,c}, \quad \bar{X} \leftrightarrow X.$$

Define maps

$$\tilde{f}_i^c : B(\mathbf{v}-c\mathbf{e}^i,\infty)_{i,0} \to B(\mathbf{v},\infty)_{i,c}, \quad \tilde{f}_i^c(\bar{X}) = X,$$
$$\tilde{e}_i^c : B(\mathbf{v},\infty)_{i,c} \to B(\mathbf{v}-c\mathbf{e}^i,\infty)_{i,0}, \quad \tilde{e}_i^c(X) = \bar{X},$$

and then define

$$\tilde{e}_i, \tilde{f}_i : B_g(\infty) \to B_g(\infty) \sqcup \{0\}$$

by

$$\tilde{e}_i^c : B(\mathbf{v},\infty)_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v}-c\mathbf{e}^i,\infty)_{i,0} \xrightarrow{\tilde{f}_i^{c-1}} B(\mathbf{v}-\mathbf{e}^i,\infty)_{i,c-1}, \quad c>0,$$
$$\tilde{f}_i^c : B(\mathbf{v},\infty)_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v}-c\mathbf{e}^i,\infty)_{i,0} \xrightarrow{\tilde{f}_i^{c+1}} B(\mathbf{v}+\mathbf{e}^i,\infty)_{i,c+1}.$$

We set $\tilde{e}_i(X) = 0$ for $X \in B(\mathbf{v},\infty)_{i,0}$. We also define

$$\mathrm{wt} : B_g(\infty) \to P, \quad \mathrm{wt}(X) = -\sum_{i \in Q_0} v_i\alpha_i \text{ for } X \in B(\mathbf{v},\infty),$$
$$\varphi_i(X) = \varepsilon_i(X) + \langle h_i, \mathrm{wt}(X)\rangle.$$

Here $h_i = [e_i, f_i]$ is the usual element of the Cartan subalgebra of $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathfrak{h}$ with $\mathfrak{h}^*$. The following theorem was proved by Kashiwara and Saito.

Theorem 3.5 ([8, Theorem 5.3.2]) *The definitions above endow $B_g(\infty)$ with the structure of a $\mathfrak{g}$-crystal and $B_g(\infty)$ is isomorphic to $B(\infty)$, the crystal corresponding to the lower half of the quantized enveloping algebra, $U_q^-(\mathfrak{g})$.*

4 The lagrangian Nakajima quiver variety

The goal in this section is to modify the definition of the Lusztig quiver variety to obtain varieties giving a geometric realization of the crystals of irreducible integrable highest weight representations. To motivate the definitions, we first recall the construction of these representations via Verma modules.

As before, we let $\mathfrak{g}$ be a Kac-Moody algebra with symmetric Cartan matrix, consider a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

and set $\mathfrak{b}^{\pm} = \mathfrak{h} \oplus \mathfrak{n}^{\pm}$. We fix a highest weight $\lambda \in P^+$, where P^+ is the dominant weight lattice. Then we define a $U(\mathfrak{b}^+)$-module $\mathbb{C}_\lambda = \mathbb{C}v_\lambda$ by

$$\begin{aligned} h \cdot v_\lambda &= \lambda(h)v_\lambda, \quad h \in \mathfrak{h}, \\ x \cdot v_\lambda &= 0, \quad x \in \mathfrak{n}^+. \end{aligned}$$

The *Verma module* of highest weight λ is then defined to be the $U(\mathfrak{g})$-module

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}_\lambda,$$

with the $U(\mathfrak{g})$-action given by left multiplication on the left factor. It follows that $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ as vector spaces. There exists a unique maximal submodule $I(\lambda)$ of $M(\lambda)$ and the irreducible integrable highest weight representation of highest weight λ is defined to be the quotient $V(\lambda) := M(\lambda)/I(\lambda)$.

We summarize this construction as follows:

$$U(\mathfrak{n}^-) \overset{\text{shift weights}}{\rightsquigarrow} M(\lambda) \overset{\text{cut}}{\rightsquigarrow} V(\lambda). \tag{4.1}$$

An analogous phenomenon occurs for crystals. Namely, the crystal $B(\lambda)$ embeds into the crystal $B(\infty)$ in the following manner. For all $\lambda \in P^+$, there exists a map $\psi_\lambda : B(\lambda) \to B(\infty)$ such that

1. ψ_λ is injective,
2. $\psi_\lambda(b_\lambda) = 1$,
3. $\psi_\lambda(\tilde{f}_i b) = \tilde{f}_i \psi_\lambda(b)$ when $\tilde{f}_i(b) = 0$,
4. $\psi_\lambda(\tilde{e}_i b) = \tilde{e}_i \psi_\lambda(b)$ for all $b \in B(\lambda)$, and
5. $\operatorname{wt} \psi_\lambda(b) = \operatorname{wt} b - \lambda$, $\varepsilon_i(\psi_\lambda(b)) = \varepsilon_i(b)$ for all $b \in B(\lambda)$.

Property (5) is analogous to the "shifting of weights" in (4.1) and the fact that some $\tilde{f}_i$'s act as zero in $B(\lambda)$ (while this does not occur in $B(\infty)$) corresponds to the "cutting" in (4.1). We now aim to mimic these procedures using quiver varieties.

Let $Q = (Q_0, Q_1)$ be the double quiver corresponding to our Kac-Moody algebra $\mathfrak{g}$ and choose an orientation Ω. Fix $\mathbf{v}, \mathbf{w} \in \mathbb{N}^{Q_0}$ and let V and W be Q_0-graded vector spaces of graded dimensions $\mathbf{v}$ and $\mathbf{w}$ respectively. The graded dimension $\mathbf{w}$ will correspond to the highest weight

$$\omega_w := \textstyle\sum_{i \in Q_0} w_i \omega_i \tag{4.2}$$

of the representation whose crystal we would like to construct (here ω_i is the ith fundamental weight of $\mathfrak{g}$). The graded dimension $\mathbf{v}$ corresponds to a weight space in that representation. More specifically, it corresponds to the weight space of weight

$$\omega_w - \alpha_v, \text{ where } \alpha_v = \textstyle\sum_{i \in Q_0} v_i \alpha_i. \tag{4.3}$$

Here α_i is the ith simple root of $\mathfrak{g}$.

Define

$$\Lambda(\mathbf{v}, \mathbf{w}) := \Lambda(\mathbf{v}) \times \textstyle\bigoplus_{i \in Q_0} \operatorname{Hom}(V_i, W_i).$$

See Figure 3. The space $\operatorname{Hom}(V_i, W_i)$ is affine and hence we have a bijection

$$\text{Irreducible components of } \Lambda(\mathbf{v}, \mathbf{w}) \overset{\text{1-1}}{\longleftrightarrow} \text{Irreducible components of } \Lambda(\mathbf{v}).$$

We think of the passage from $\Lambda(\mathbf{v})$ to $\Lambda(\mathbf{v}, \mathbf{w})$ as the analogue of the "shifting weights" procedure in (4.1).

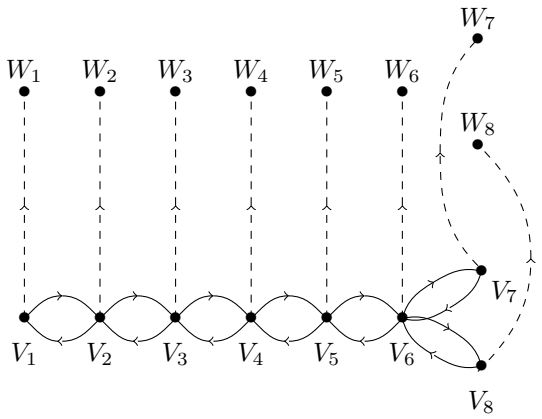


Figure 3 The linear maps involved in the definition of the space $\Lambda(\mathbf{v}, \mathbf{w})$ when $\mathfrak{g} = \mathfrak{so}_{16}$, the simple Lie algebra of type D_8.

Definition 4.1 (Stability condition) We say a point $(x, t) \in \Lambda(\mathbf{v}, \mathbf{w})$ is *stable* if the following condition holds: If S is an x-invariant Q_0-graded subspace of V such that $t_i(S_i) = 0$ for all $i \in Q_0$, then $S_i = 0$ for all $i \in Q_0$. We denote the set of stable points in $\Lambda(\mathbf{v}, \mathbf{w})$ by $\Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}}$.

The group G_V acts on $\Lambda(\mathbf{v}, \mathbf{w})$ as

$$(g, (x, t)) = (x', t'), \text{ where } x'_a = g_{t(a)} x_a g_{s(a)}^{-1}, \ t'_i = t_i g_i^{-1}.$$

Lemma 4.2 ([11, Lemma 3.10]) *The stabilizer of any point of* $\Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}}$ *in* G_V *is trivial.*

Definition 4.3 (Lagrangian Nakajima quiver variety) We define

$$\mathfrak{L}(\mathbf{v}, \mathbf{w}) = \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}} / G_V,$$

and we call $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ a *lagrangian Nakajima quiver variety.*

Remarks 4.4

1. The name lagrangian Nakajima quiver variety arises from the fact that the varieties defined here are lagrangian subvarieties of what are called (smooth) Nakajima quiver varieties. See [11, 12] for details.
2. For a given irreducible component X of $\Lambda(\mathbf{v}, \mathbf{w})$, one of the following two conditions holds:
   (a) No point of X satisfies the stability condition.
   (b) All points in an open dense subset of X satisfy the stability condition.
   Thus we speak of each irreducible component as failing or satisfying the stability condition in case the first or second condition (respectively) above holds. Imposing the stability condition corresponds to the "cutting" procedure in (4.1).
3. Since G_V acts freely on $\Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}}$, there is a bijection

Irreducible components of $\Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}} \overset{\text{1-1}}{\longleftrightarrow}$ Irreducible components of $\mathfrak{L}(\mathbf{v}, \mathbf{w})$.

We summarize the geometric construction as follows (compare to (4.1)):

$$\Lambda(\mathbf{v}) \overset{\text{shift weights}}{\rightsquigarrow} \Lambda(\mathbf{v}) \times \bigoplus\nolimits_{i \in Q_0} \mathrm{Hom}(V_i, W_i) \overset{\text{cut}}{\rightsquigarrow} \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}} \rightsquigarrow \mathfrak{L}(\mathbf{v}, \mathbf{w}).$$

Example 4.5 Let $\mathfrak{g} = \mathfrak{sl}_2$. In this case Q is the quiver consisting of a single vertex and no edges. Then $\mathbf{v} = v$ and $\mathbf{w} = w$ are just natural numbers and

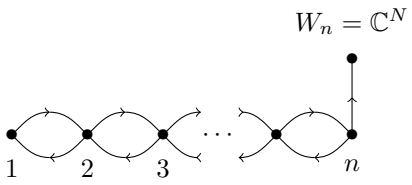
$$\Lambda(v,w)^{\mathrm{st}} = \{t : V \to W \mid t \text{ injective}\},$$

which is empty unless $v \leq w$. If this condition is satisfied, then

$$\mathfrak{L}(v,w) = \Lambda(v,w)^{\mathrm{st}}/GL(V) \cong \mathrm{Gr}(v,w),$$

where $\mathrm{Gr}(v,w)$ is the grassmannian of v-dimensional subspaces of $\mathbb{C}^w$ and the isomorphism is induced by the map $t \mapsto \operatorname{im} t$.

Exercise 4.6 Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\mathbf{w} = N\mathbf{e}^n$ for some $N \in \mathbb{N}$. Then we can picture the maps involved in the quiver variety as follows:

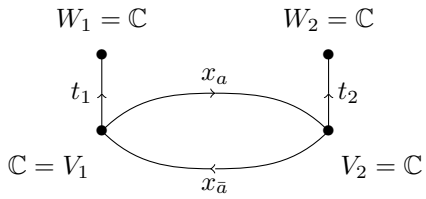


Show that $\mathfrak{L}(v,w)$ is empty unless $v_1 \leq v_2 \leq \cdots \leq v_n \leq N$, in which case

$$\mathfrak{L}(\mathbf{v},\mathbf{w}) \cong \{0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq \mathbb{C}^N \mid \dim V_i = v_i\},$$
$$(x,t) \mapsto (V_i), \text{ where } V_i = \operatorname{im} t a_{n-1\to n} a_{n-2\to n-1} \cdots a_{i\to i+1}.$$

Here $a_{j\to j+1}$ is the arrow from vertex j to vertex $j+1$ for $1 \leq j \leq n-1$. Thus $\mathfrak{L}(\mathbf{v},\mathbf{w})$ is the partial flag variety. See [6, Section 4.1].

Example 4.7 Let $\mathfrak{g} = \mathfrak{sl}_3$, $\mathbf{w} = \mathbf{e}^1 + \mathbf{e}^2$, and $\mathbf{v} = \mathbf{e}^1 + \mathbf{e}^2$. This corresponds, under the identifications (4.2) and (4.3), to the weight zero subspace of the adjoint representation of $\mathfrak{sl}_3$ (which is two-dimensional). We can picture the maps involved in our quiver varieties as follows:



We split the points of the quiver variety into two cases. We first consider the case where $x_a = 0$. Then the stability condition implies that

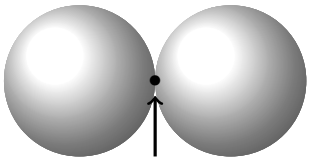
$$\ker x_{\bar{a}} \cap \ker t_2 \neq 0, \quad t_1 \neq 0.$$

There is a unique element of $GL(V_1)$ whose action changes t_1 to the identity map. Therefore, we can describe the quotient defining the subvariety of $\mathfrak{L}(\mathbf{v},\mathbf{w})$ given by $x_a = 0$ by fixing $t_1 = 1$ and considering only the quotient by $GL(V_2)$. Thus we can identify this subvariety of the quiver variety with

$$\{(x_{\bar{a}}, t_2) \in \mathbb{C}^2 \mid (x_{\bar{a}}, t_2) \neq (0,0)\}/\mathbb{C} \cong \mathbb{P}^1,$$

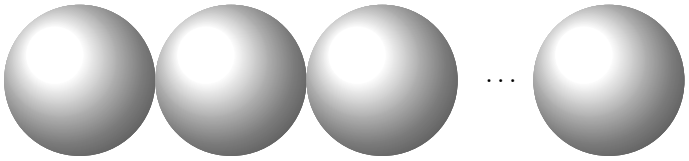
where the quotient by $\mathbb{C}$ is the quotient by $GL(V_2)$. The second case, where $x_{\bar{a}} = 0$ is analogous and also yields $\mathbb{P}^1$. Therefore, the lagrangian Nakajima quiver variety

$\mathfrak{L}(\mathbf{v},\mathbf{w})$ consists of two projective lines meeting at a point (the point where x_a and $x_{\bar{a}}$ are both zero).



$x_a = 0,\ x_{\bar{a}} = 0$

Exercise 4.8 Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathbf{w} = \mathbf{e}^1 + \mathbf{e}^n$ and $\mathbf{v} = \mathbf{e}^1 + \mathbf{e}^2 + \cdots + \mathbf{e}^n$. Under the identifications (4.2) and (4.3), this corresponds to the n-dimensional weight zero subspace of the adjoint representation of $\mathfrak{sl}_{n+1}$. Show that the lagrangian Nakajima quiver variety is a union of n projective lines as shown below.



Having defined the lagrangian Nakajima quiver varieties, our goal is to describe the structure of a crystal on the set of irreducible components of $\bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v},\mathbf{w})$. For $c \in \mathbb{N}$, define

$$\mathfrak{F}(\mathbf{v},\mathbf{w};c\mathbf{e}^i) = \{(x,t,S) \mid (x,t) \in \Lambda(\mathbf{v},\mathbf{w})^{\mathrm{st}},\ S \text{ an } x\text{-invariant } Q_0\text{-graded subspace of } V,\ \dim S = \mathbf{v} - c\mathbf{e}^i\}/GL_V.$$

We then have natural projections

$$\mathfrak{L}(\mathbf{v} - c\mathbf{e}^i,\mathbf{w}) \xleftarrow{\pi_1} \mathfrak{F}(\mathbf{v},\mathbf{w};c\mathbf{e}^i) \xrightarrow{\pi_2} \mathfrak{L}(\mathbf{v},\mathbf{w}),$$

where $\pi_1(G_V \cdot (x,t)) = G_S \cdot (x|_S, t|_S)$ and $\pi_2(G_V \cdot (x,t,S)) = G_V \cdot (x,t)$. For $c \in \mathbb{N}$, define

$$\varepsilon_i : \mathfrak{L}(\mathbf{v},\mathbf{w}) \to \mathbb{N},\ \varepsilon_i(x,t) = \dim \mathrm{Coker}\left(\bigoplus_{a,\, t(a)=i} V_{s(a)} \to V_i\right).$$

For $i \in Q_0$ and $c \in \mathbb{N}$, let

$$\mathfrak{L}(\mathbf{v},\mathbf{w})_{i,c} = \{G_V \cdot (x,t) \in \mathfrak{L}(\mathbf{v},\mathbf{w}) \mid \varepsilon_i((x,t)) = c\}.$$

Then $\mathfrak{L}(\mathbf{v},\mathbf{w})_{i,c}$ is a locally closed subvariety of $\mathfrak{L}(\mathbf{v},\mathbf{w})$. If $\mathfrak{L}(\mathbf{v},\mathbf{w})_{i,c} \neq \emptyset$, then

$$\pi_1^{-1}(\mathfrak{L}(\mathbf{v} - c\mathbf{e}^i,\mathbf{w})_{i,0}) = \pi_2^{-1}(\mathfrak{L}(\mathbf{v},\mathbf{w})_{i,c}).$$

Setting

$$\mathfrak{F}(\mathbf{v},\mathbf{w};c\mathbf{e}^i)_{i,0} = \pi_1^{-1}(\mathfrak{L}(\mathbf{v} - c\mathbf{e}^i,\mathbf{w})_{i,0}) = \pi_2^{-1}(\mathfrak{L}(\mathbf{v},\mathbf{w})_{i,c}),$$

we have the natural projections

$$\mathfrak{L}(\mathbf{v} - c\mathbf{e}^i,\mathbf{w})_{i,0} \xleftarrow{\pi_1} \mathfrak{F}(\mathbf{v},\mathbf{w};c\mathbf{e}^i)_{i,0} \xrightarrow{\pi_2} \mathfrak{L}(\mathbf{v},\mathbf{w})_{i,c}.$$

The restriction of π_2 to $\mathfrak{F}(\mathbf{v},\mathbf{w};c\mathbf{e}^i)_{i,0}$ is an isomorphism and $\mathfrak{L}(\mathbf{v} - c\mathbf{e}^i,\mathbf{w})_{i,0}$ is an open subvariety of $\mathfrak{L}(\mathbf{v} - c\mathbf{e}^i,\mathbf{w})$.

Lemma 4.9 ([14, Lemma 4.2.2]) 1. *For $i \in Q_0$,*

$$\mathfrak{L}(\mathbf{0}, \mathbf{w})_{i,c} = \begin{cases} \mathrm{pt} & \text{if } c = 0, \\ \emptyset & \text{if } c > 0. \end{cases}$$

2. *If $\mathfrak{L}(\mathbf{v}, \mathbf{w})_{i,c} \neq \emptyset$, then the fiber of the restriction of π_1 to $\mathfrak{F}(\mathbf{v}, \mathbf{w}; c\mathbf{e}^i)_{i,0}$ is isomorphic to a grassmannian variety.*

Corollary 4.10 *If $\mathfrak{L}(\mathbf{v}, \mathbf{w})_{i,c} \neq \emptyset$, then there is a one-to-one correspondence between the set of irreducible components of $\mathfrak{L}(\mathbf{v}-c\mathbf{e}^i, \mathbf{w})_{i,0}$ and the set of irreducible components of $\mathfrak{L}(\mathbf{v}, \mathbf{w})_{i,c}$.*

Let $B(\mathbf{v}, \mathbf{w})$ be the set of irreducible components of $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ and $B_g(\mathbf{w}) = \bigsqcup_{\mathbf{v}} B(\mathbf{v}, \mathbf{w})$. As in Section 3, for $X \in B(\mathbf{v}, \mathbf{w})$, let $\varepsilon_i(X) = \varepsilon_i((x, t))$ for a generic point $G_V \cdot (x, t) \in X$. Then for $c \in \mathbb{N}$, define

$$B(\mathbf{v}, \mathbf{w})_{i,c} = \{X \in B(\mathbf{v}, \mathbf{w}) \mid \varepsilon_i(X) = c\}.$$

Thus, by Corollary 4.10 we have a bijective correspondence

$$B(\mathbf{v} - c\mathbf{e}^i, \mathbf{w})_{i,0} \cong B(\mathbf{v}, \mathbf{w})_{i,c}, \quad \bar{X} \leftrightarrow X.$$

We then define maps

$$\tilde{f}_i^c : B(\mathbf{v} - c\mathbf{e}^i, \mathbf{w})_{i,0} \to B(\mathbf{v}, \mathbf{w})_{i,c}, \quad \tilde{f}_i^c(\bar{X}) = X,$$
$$\tilde{e}_i^c : B(\mathbf{v}, \mathbf{w})_{i,c} \to B(\mathbf{v} - c\mathbf{e}^i, \mathbf{w})_{i,0}, \quad \tilde{e}_i^c(X) = \bar{X}.$$

Then we define

$$\tilde{e}_i,\ \tilde{f}_i : B(\mathbf{w}) \to B(\mathbf{w}) \sqcup \{0\}$$

by

$$\tilde{e}_i : B(\mathbf{v}, \mathbf{w})_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v} - c\mathbf{e}^i, \mathbf{w})_{i,0} \xrightarrow{\tilde{f}_i^{c-1}} B(\mathbf{v} - \mathbf{e}^i, \mathbf{w})_{i,c-1},$$
$$\tilde{f}_i : B(\mathbf{v}, \mathbf{w})_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v} - c\mathbf{e}^i, \mathbf{w})_{i,0} \xrightarrow{\tilde{f}_i^{c+1}} B(\mathbf{v} + \mathbf{e}^i, \mathbf{w})_{i,c+1}.$$

We set $\tilde{e}_i(X) = 0$ for $X \in B(\mathbf{v}, \mathbf{w})_{i,0}$ and $\tilde{f}_i(X) = 0$ for $X \in B(\mathbf{v}, \mathbf{w})_{i,c}$ with $B(\mathbf{v}, \mathbf{w})_{i,c+1} = \emptyset$. Furthermore, we define

$$\mathrm{wt} : B_g(\mathbf{w}) \to P, \quad \mathrm{wt}\, X = \omega_{\mathbf{w}} - \alpha_{\mathbf{v}} \text{ for } X \in B(\mathbf{v}, \mathbf{w}),$$
$$\varphi_i(X) = \varepsilon_i(X) + \langle h_i, \mathrm{wt}\, X \rangle.$$

The following theorem was proven by Saito.

Theorem 4.11 ([14, Theorem 4.6.4]) *The definitions above endow $B_g(\mathbf{w})$ with the structure of a $\mathfrak{g}$-crystal and $B_g(\mathbf{w})$ is isomorphic to the crystal $B(\omega_{\mathbf{w}})$ of the irreducible integrable highest weight representation of highest weight $\omega_{\mathbf{w}}$.*

Remarks 4.12 1. While Theorem 4.11 only applies to crystals with symmetric Cartan matrix, one can use a "folding" procedure to produce a geometric construction of the crystals of highest weight representations in arbitrary symmetrizable type (see [15]).

2. Nakajima quiver varieties can be used to construct the full structure (as opposed to the crystal structure) of the irreducible integrable highest weight representations of a Kac-Moody algebra $\mathfrak{g}$ with symmetric Cartan matrix. The procedure is similar to the one described in [6, Section 4.3]. One defines a smooth quiver variety $\mathfrak{M}$ and a singular quiver variety $\mathfrak{M}_0$ and there exists a resolution of singularities $\pi : \mathfrak{M} \to \mathfrak{M}_0$. Then $\mathfrak{L} = \pi^{-1}(0)$ and we define

$Z = \mathfrak{M} \times_\pi \mathfrak{M}$. Then convolution gives the homology of Z the structure of an algebra (isomorphic to a quotient of $U(\mathfrak{g})$) and defines an action of this algebra on the homology of $\mathfrak{L}$, realizing the representation in question. For details, see [11, 12].

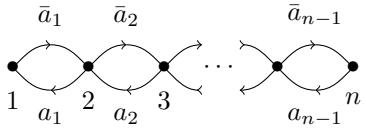
3. In type A (i.e. when $\mathfrak{g} = \mathfrak{sl}_n$), the quiver variety construction and the Ginzburg construction described in [6, Section 4] are closely related but slightly different. See [17] for details.
4. One can define quiver varieties which yield geometric realizations of tensor products of crystals of highest weight representations (see [9, 10, 13]).

5 Connections to combinatorial realizations of crystal graphs

In this section, we will describe the precise relationship between the combinatorial realizations of crystals given by tableaux and the geometric realizations introduced in Section 4. For this section, we fix $\mathfrak{g} = \mathfrak{sl}_{n+1}$ to be the simple Lie algebra of type A_n.

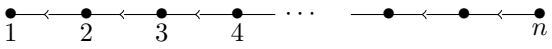
Recall that the double quiver Q associated to $\mathfrak{g}$ is as follows.

Q



We choose the orientation $\Omega = \{a_1, a_2, \ldots, a_{n-1}\}$ consisting of the left-pointing arrows and let Q_Ω be the corresponding quiver containing only the arrows in Ω:

Q_Ω

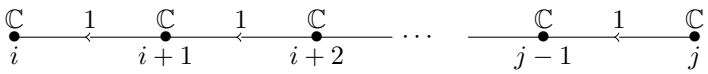


Let $V = \bigoplus_{i \in I} V_i$ be a Q_0-graded vector space of graded dimension $\mathbf{dim}\, V = \mathbf{v}$. By Gabriel's Theorem (Theorem 2.6) and the discussion in Section 2, we have the following bijections.

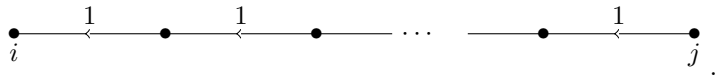
G_V-orbits in $E_{V,\Omega}$	$\overset{\text{1-1}}{\longleftrightarrow}$	Isom. classes of reps. of Q_Ω	$\overset{\text{1-1}}{\longleftrightarrow}$	Collections of positive roots of $\mathfrak{g}$ adding to $\sum_i v_i \alpha_i$
		Isom. classes of indecomposable reps.	$\overset{\text{1-1}}{\longleftrightarrow}$	positive roots of $\mathfrak{g}$

Recall that under these bijections, the positive root $\alpha_i + \cdots + \alpha_j$, $i \le j$, corresponds to the isomorphism class of the representation $(V^{i,j}, x^{i,j})$ of Example 2.5. We depict this representation by the diagram

$V^{i,j}$



or simply by

$V^{i,j}$ 

In such pictures, each vertex represents a basis vector in the representation, and we identify the two. Vertex labels (elements of Q_0) indicate the degree in which this vector lives and arrow labels indicate the action of x ($= x^{i,j}$ here). Thus an arrow labeled by $z \in \mathbb{C}$ from a vertex a to a vertex b indicates that the coefficient of b in the expansion of $x(a)$ as a linear combination of the vertices is z. We will always vertically line up vertices of the same degree and degrees will increase from left to right.

Recall that in finite type, the nilpotency condition in the definition of the Lusztig quiver varieties is superfluous and thus

$$\Lambda(\mathbf{v}) = \{x \in E_V = E_{V,\Omega} \oplus E_{V,\bar{\Omega}} \mid \psi(x) = 0\},$$

where $\psi(x) = 0$ is the moment map condition. In our case, the moment map condition is equivalent to the following set of conditions:

$$x_{a_i} x_{\bar{a}_i} = x_{\bar{a}_{i-1}} x_{a_{i-1}}, \quad i \neq 1, n, \tag{5.1}$$

$$x_{a_1} x_{\bar{a}_1} = 0, \tag{5.2}$$

$$x_{\bar{a}_{n-1}} x_{a_{n-1}} = 0. \tag{5.3}$$

We adopt the convention that $x_{a_i} = 0$ and $x_{\bar{a}_i} = 0$ for $i \leq 0$ or $i \geq n$.

We know that the irreducible components of $\Lambda(\mathbf{v})$ are the closures of the conormal bundles to the orbits in $E_{V,\Omega}$ and that $E_{V,\bar{\Omega}}$ corresponds to the cotangent direction. Suppose we consider an orbit corresponding to a single positive root. More precisely, we consider the orbit through the point $(V^{i,j}, x^{i,j})$ for some $i \leq j$. We would like to describe the conormal bundle to this orbit. Because the group G_V acts transitively on the orbit and identifies the fibers over points, it suffices to describe the fiber of the conormal bundle over a particular point, namely $(V^{i,j}, x^{i,j})$. We have seen that this fiber is given by the set

$$\{x = (x^{i,j}, x_{\bar{\Omega}} = (x_{\bar{a}_k})_{1 \leq k \leq n-1}) \in E_V \mid \psi(x) = 0\}.$$

Suppose $x = (x^{i,j}, x_{\bar{\Omega}} = (x_{\bar{a}_k})_{1 \leq k \leq n-1})$ is in the fiber. Since $V^{i,j}_k = 0$ unless $i \leq k \leq j$, we have $x_{\bar{a}_k} = 0$ unless $i \leq k \leq j-1$. Therefore, by (5.1) (if $i > 1$) or (5.2) (if $i = 1$), we have $x_{a_i} x_{\bar{a}_i} = 0$. But $x_{a_i} = 1$ and so $x_{\bar{a}_i} = 0$. Again, one can use (5.1) to show that $x_{\bar{a}_{i+1}} = 0$. Continuing in this manner, we see that in fact $x_{\bar{\Omega}} = 0$. Therefore, the conormal bundle to the orbit is just the orbit itself (i.e. each fiber in the conormal bundle consists of a single point).

Now consider an orbit corresponding to 2 positive roots. More precisely, we consider the orbit through the point $x^{i_1,j_1} \oplus x^{i_2,j_2}$ for some $i_1 \leq j_1$ and $i_2 \leq j_2$. We picture this representation as in Figure 4.

We now wish to describe the conormal bundle to this orbit. Again, it suffices to describe the fiber of the conormal bundle over a particular point, namely $(V^{i_1,j_1} \oplus V^{i_2,j_2}, x^{i_1,j_1} \oplus x^{i_2,j_2})$, and this fiber is given by the set

$$\{x = (x^{i_1,j_1} \oplus x^{i_2,j_2}, x_{\bar{\Omega}}) \in E_V \mid \psi(x) = 0\}.$$

For $l = 1, 2$, let $V^{i_l,j_l}_k = \mathbb{C}v^l_k$, $i_l \leq k \leq j_l$, such that $x^{i,j}_{a_k}(v^l_{k+1}) = v^l_k$ for $i_l \leq k \leq j_l - 1$. Set $v^l_k = 0$ for $k < i_l$ or $k > j_l$. What are the possible values of $x_{\bar{a}_k}(v^2_k)$ for $i_2 \leq k \leq j_2$? In general, $x_{\bar{a}_k}(v^2_k)$ is a linear combination of v^1_{k+1} and v^2_{k+1}.

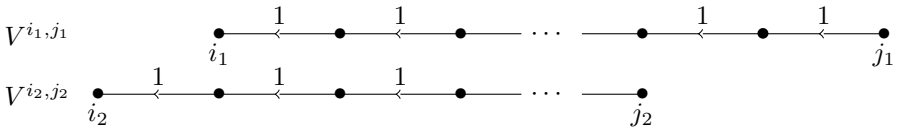


Figure 4 The quiver representation $x^{i_1,j_1} \oplus x^{i_2,j_2}$. Each vertex represents a basis vector in $V^{i_1,j_1} \oplus V^{i_2,j_2}$. Vertex labels indicate the degree of the corresponding basis vector and vertices of the same degree are vertically aligned. Edges indicate the action of $x^{i_1,j_1} \oplus x^{i_2,j_2}$ with coefficients given by edge labels.

However, by the same argument as above, one can show that the coefficient of v^2_{k+1} must in fact be zero. Therefore

$$x_{\bar{a}_k}(v^2_k) = c_k v^1_{k+1},$$

for some $c_k \in \mathbb{C}$. Suppose that $i_1 \leq k+1 \leq j_1$ and $c_k \neq 0$. Then if $k+1 > j_2$, by (5.1) we have

$$x_{\bar{a}_{k-1}}(v^2_{k-1}) = x_{\bar{a}_{k-1}} x_{a_{k-1}}(v^2_k) = x_{a_k} x_{\bar{a}_k}(v^2_k) = c_k x_{a_k}(v^1_{k+1}) = c_k v^1_k \neq 0.$$

Continuing in this manner, we see that $x_{\bar{a}_{i_1-1}}(v^2_{i_1-1}) \neq 0$ and thus $i_2 < i_1$.

Now, if $k+1 \leq j_2$, then

$$x_{a_{k+1}} x_{\bar{a}_{k+1}}(v^2_{k+1}) = x_{\bar{a}_k} x_{a_k}(v^2_{k+1}) = x_{\bar{a}_k}(v^2_k) \neq 0.$$

Therefore $x_{\bar{a}_{k+1}}(v^2_{k+1}) \neq 0$. But $x_{\bar{a}_{k+1}}(v^2_{k+1})$ must be a multiple of v^1_{k+2} as above. Thus, we must have $k+2 \leq j_1$. Continuing in this way, we see that $j_2 < j_1$. Therefore, in order for $x_{\bar{a}_k}(v^2_k)$ to be nonzero for any k, we must have $i_2 < i_1$ and $j_2 < j_1$ (for instance, we could have the situation pictured in Figure 4).

Now, let $x = (x_\Omega, x_{\bar{\Omega}})$ lie in the conormal bundle to the point

$$x_\Omega = \bigoplus_{l=1}^{s} x^{i_l,j_l}.$$

By reordering the indices if necessary, we can assume that $i_1 \geq i_2 \geq \cdots \geq i_s$. As above, let $V^{i_l,j_l}_k = \mathbb{C}v^l_k$, for $1 \leq l \leq s$. By the above arguments, $x_{\bar{a}_k}(v^l_k)$ must be a linear combination of $\{v^a_{k+1}\}_{a<l}$. Thus,

$$v^1_{i_1} \in \ker x_{a_{i_1-1}} \cap \ker x_{\bar{a}_{i_1}}. \tag{5.4}$$

Exercise 5.1 Show that a point $(x,t) \in \Lambda(\mathbf{v},\mathbf{w})$ satisfies the stability condition if and only if

$$\ker x_{a_{k-1}} \cap \ker x_{\bar{a}_k} \cap \ker t_k = 0 \quad \forall\ 1 \leq k \leq n.$$

Now consider the lagrangian Nakajima quiver variety $\mathfrak{L}(\mathbf{v},\mathbf{w})$ for $\mathbf{w} = \mathbf{e}^r$ for some $1 \leq r \leq n$. Suppose a point (x,t) satisfies the stability condition where x is in the conormal bundle to the point $x_\Omega = \bigoplus_{l=1}^{s} x^{i_l,j_l}$. By Exercice 5.1 and (5.4), we must have $i_1 = r$ and there can be no other v^l_k in $\ker x_{a_{k-1}} \cap \ker x_{\bar{a}_k}$ for any k. By the above considerations, $v^l_{i_l}$ is in $\ker x_{a_{i_l-1}} \cap \ker x_{\bar{a}_{i_l}}$ unless $i_l + 1 = i_{l-1}$. Thus we have $i_{l+1} = i_l - 1$ and $x_{\bar{a}_{i_{l+1}}}\left(v^{l+1}_{i_{l+1}}\right) = c_l v^l_{i_l} \neq 0$ for $1 \leq l \leq s-1$. Thus, by the above, we must have $j_{l+1} < j_l$ for $1 \leq l \leq s-1$. Thus, the element x can be depicted as in Figure 5.

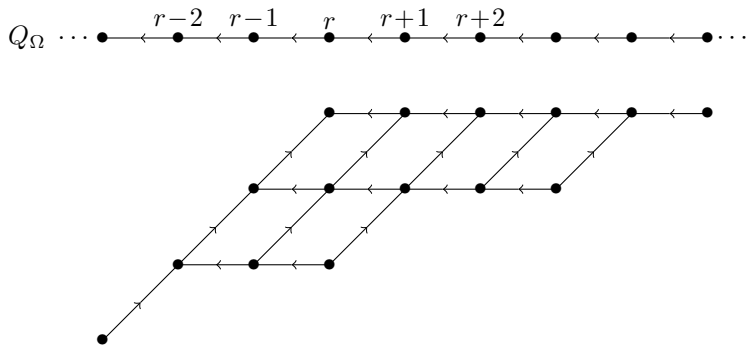


Figure 5 A depiction of an element x of a point $(x,t) \in \Lambda(\mathbf{v},\mathbf{e}^r)$ satisfying the stability condition.

In the tableaux realization of $B(\Lambda_k)$ (see [5]), the vertices of the crystal graph consist of single column semistandard tableaux with k rows and entries from the set $\{1,2,\ldots,n+1\}$. Note that there is a one-to-one correspondence between this set of tableaux and the set of Young diagrams with at most k rows and whose rows have at most $n+1-k$ boxes. Precisely, the semistandard tableau

$$\begin{array}{|c|}\hline n_1 \\ \hline n_2 \\ \hline \vdots \\ \hline n_k \\ \hline \end{array} \tag{5.5}$$

corresponds to the Young diagram (or partition) $(n_k-k, n_{k-1}-1, \ldots, n_1-1)$. It follows that we have a bijective correspondence between $B(\omega_k)$ and the set of conormal bundles satisfying the stability condition, where the tableaux (5.5) corresponds to the closure of the conormal bundle to the orbit through the representation

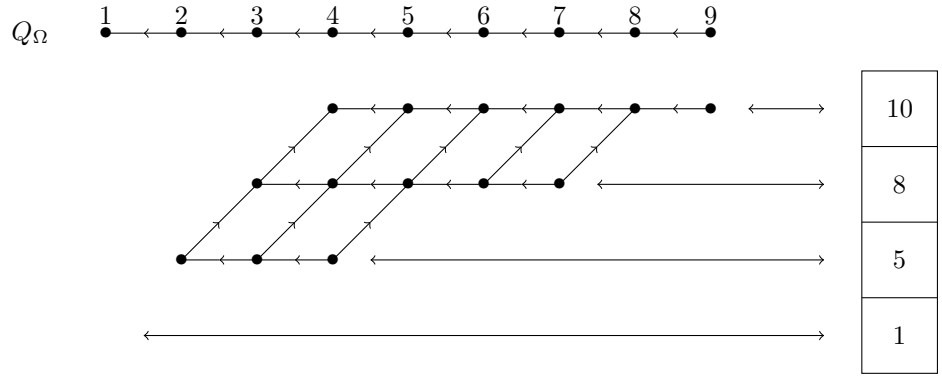
$$\left(\textstyle\bigoplus_{i=1}^k V^{k+1-i,n_i-1}, \bigoplus_{i=1}^k x^{k+1-i,n_i-1}\right).$$

Loosely speaking, we have the correspondence

$\boxed{j}$ in the ith row $\longleftrightarrow$

.

Example 5.2 Suppose $\mathfrak{g} = \mathfrak{sl}_{10}$ and $k = 4$. Then we have the following correspondence.



Here we draw the tableau upside down in order to make the correspondence clearer. The tableau

$$T = \begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline 8 \\ \hline 10 \\ \hline \end{array}$$

then corresponds to the irreducible component of the lagrangian Nakajima quiver variety obtained as follows: Let C_T be the closure of the conormal bundle to the G_V-orbit through

$$x^{4,9} \oplus x^{3,7} \oplus x^{2,4}.$$

Then T corresponds to the irreducible component

$$\left(\left(C_T \times \sum_{i=1}^{n} \mathrm{Hom}(V_i, W_i) \right) \cap \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}} \right) / G_V.$$

In fact, one can show that the Nakajima quiver varieties $\Lambda(\mathbf{e}^k)$ corresponding to fundamental weights ω_k are all single points (this can be shown directly or by using the dimension formula for Nakaijima quiver varieties and the fact that these varieties are connected).

What about more general highest weights (that is, highest weights that are not fundamental weights)? Consider the highest weight $\omega_{\mathbf{w}} = \sum_{i \in Q_0} w_i \omega_i$. Then an argument similar to the above demonstrates that we can have w_i Young diagrams associated to each vertex i. In order to avoid double counting, we must associate each irreducible component of the lagrangian Nakajima quiver variety to a unique such collection of Young diagrams. We adopt the convention that larger indecomposable representations of Q_Ω (corresponding to the rows of the Young diagrams) are associated to the Young diagrams attached to vertices of higher degree (see [16] for a more precise treatment). Each Young diagram corresponds to a single column tableaux as above and, if we organize these columns into a tableau, our convention corresponds precisely to the condition that this tableau be semistandard. We thus obtain a bijective map from the set $B_g(\mathbf{w})$ of irreducible components of the lagrangian Nakajima quiver variety to the set $B(\omega_w)$ of semistandard tableaux.

Theorem 5.3 ([16, Theorem 6.4]) *The identification of $B_g(\mathbf{w})$ with $B(\omega_w)$ described above is an isomorphism of crystals.*

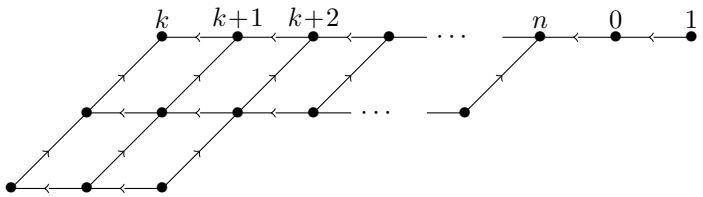


Figure 6 A diagram representing a point in the quiver variety corresponding to a level one representation (of highest weight ω_k) of the Lie algebra $\widehat{\mathfrak{sl}}_n$

Exercise 5.4 If $\mathfrak{g} = \mathfrak{sl}_3$, $\mathbf{w} = e_1 + e_2$ (so $V(\omega_{\mathbf{w}})$ is the adjoint representation) and $\mathbf{v} = e_1 + e_2$ (so $V(\omega_{\mathbf{w}})_{\omega_{\mathbf{w}} - \alpha_{\mathbf{v}}} = \mathfrak{h}$ is the zero weight space), we described the lagrangian Nakajima quiver variety $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ in Example 4.7. It consists of two projective lines meeting at a point. Show that the projective line given by $x_a = 0$ corresponds to the tableau

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

and the projective line given by $\bar{x}_a = 0$ corresponds to the tableau

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad .$$

Remarks 5.5

1. When $\mathfrak{g} = \mathfrak{so}_{2n}$ is the simple Lie algebra of type D_n, a similar explicit correspondence between the irreducible components of the lagrangian Nakajima quiver varieties and the tableaux appearing in a combinatorial realization of the irreducible highest weight representations of $\mathfrak{g}$ (see [5, § 8.5] for a description of the tableaux appearing in this model) can also be given. However, this description, as well as the tableaux appearing in the combinatorial realization, are slightly more complicated. See [16] for details.
2. When $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$ is the Lie algebra of affine type A (see Example 2.8), one can give a similar explicit enumeration of the irreducible components of the lagrangian Nakajima quiver variety. However, now the "strings" representing indecomposable representations of the quiver Q_Ω can "wrap around" the quiver and thus have arbitrary length. For level one representations, we get pictures as in Figure 6. If we straighten this picture, flip it (in the line $x = y$ if we view the picture as lying in the standard (x, y)-plane), and replace vertices by boxes, we obtain Figure 7, which is simply one of the Young walls appearing in [7, Section 7]. Since we are no longer in the finite type case, we must impose the nilpotency condition. This corresponds to the condition that the Young walls be proper.
3. In the case where $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$ and $\omega_{\mathbf{w}} = \sum_{i=0}^{n} w_i \omega_i$ is arbitrary (i.e. we consider general dominant integral weights), we get w_i Young walls associated to each vertex i as in the case of $\mathfrak{sl}_n$. Again, choosing a convention to avoid double counting becomes a condition on the heights of columns appearing. What remains are combinatorial objects called *Young pyramids*. See Figure 8 for an example of a Young pyramid. Further details on these objects and their connections to quiver varieties can be found in [15].

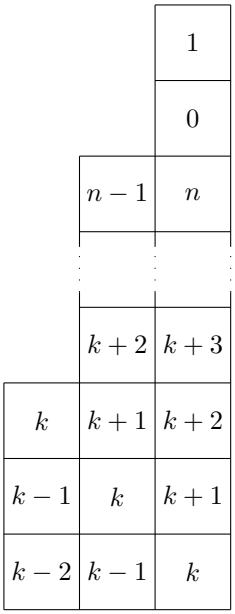


Figure 7 The Young wall corresponding to the quiver representation of Figure 6.

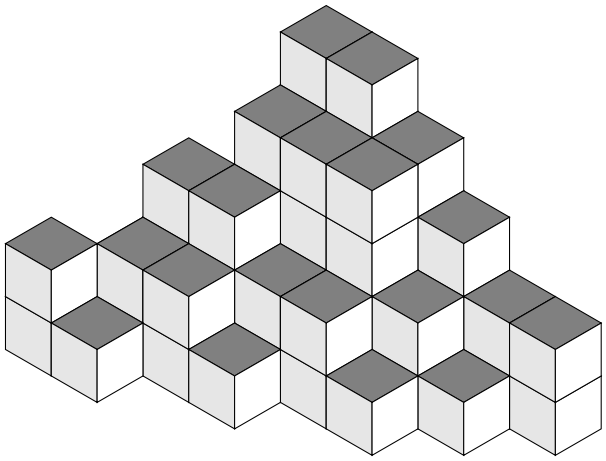


Figure 8 A Young pyramid corresponding to the highest weight $2\omega_0 + 2\omega_1 + \omega_2$. We think of each vertical wall running diagonally from top-left to bottom-right as a Young diagram or Young wall. The positions of the leftmost columns of these walls are determined by the highest weight. Here the leftmost columns of the back two walls are aligned (corresponding to the summand of $2\omega_0$), the next two walls are aligned one position to the left (corresponding to the summand of $2\omega_1$) and the frontmost wall starts one position further to the left (corresponding to the summand ω_2).

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